

# ELLIPTIC SYSTEMS OF PSEUDODIFFERENTIAL EQUATIONS IN A REFINED SCALE ON A CLOSED MANIFOLD

VLADIMIR A. MIKHAILOTS, ALEXANDR A. MURACH

**ABSTRACT.** We study a system of pseudodifferential equations that is elliptic in the sense of Petrovskii on a closed compact smooth manifold. We prove that the operator generated by the system is Fredholm one on a refined two-sided scale of the functional Hilbert spaces. Elements of this scale are the special isotropic spaces of Hörmander–Volevich–Paneah.

## 1. INTRODUCTION

In this article we consider an elliptic in Petrovskii's sense system of linear pseudodifferential equations on a closed smooth manifold. It is well known (see e.g. [1, 2]) that the operator  $A$  corresponding to this system is bounded and Fredholm in appropriate pairs of the Sobolev spaces. We investigate this operator in the Hilbert scale of the special isotropic Hörmander–Volevich–Paneah spaces [3–6]

$$H^{s,\varphi} := H_2^{(\cdot)^s \varphi(\cdot)}, \quad \langle \xi \rangle := (1 + |\xi|^2)^{1/2}. \quad (1)$$

Here,  $s \in \mathbb{R}$  and  $\varphi$  is a functional parameter slowly varying at  $+\infty$  in Karamata's sense. In particular, every standard function

$$\varphi(t) = (\log t)^{r_1} (\log \log t)^{r_2} \dots (\log \dots \log t)^{r_n}, \quad \{r_1, r_2, \dots, r_n\} \subset \mathbb{R}, \quad n \in \mathbb{N},$$

is admissible. This scale was introduced and investigated by the authors in [7, 8]. It contains Sobolev's scale  $\{H^s\} \equiv \{H^{s,1}\}$  and is attached to it by the number parameter  $s$  and being considerably finer.

Spaces of form (1) arise naturally in different spectral problems: convergence of spectral expansions of self-adjoint elliptic operators almost everywhere, in the norm of the spaces  $L_p$  with  $p > 2$  or  $C$  (see survey [9]); spectral asymptotics of general self-adjoint elliptic operators in a bounded domain, the Weyl formula, a sharp estimate of the remainder in it (see [10, 11]) and others. They may be expected to be useful in other "fine" questions. Due to their interpolation properties, the spaces  $H^{s,\varphi}$  occupy a special position among the spaces of a generalized smoothness which are actively investigated and used today (see survey [12], recent articles [13, 14] and the bibliography given there).

The main result of this article is the theorem on boundedness and the Fredholm property of the operator  $A$  in scale (1). The refined local smoothness of a solution of the elliptic system is obtained as a significant application. Also some auxiliary results which may be of interest

---

*Date:* 14/11/2007.

*2000 Mathematics Subject Classification.* Primary 35J45, Secondary 46E35.

*Key words and phrases.* Elliptic system, pseudodifferential operator, regularly varying function, scale of spaces, the Hörmander spaces, the Fredholm property.

by themselves are given. The case of scalar differential operators was investigated earlier in [8, 15–18].

## 2. THE STATEMENT OF THE PROBLEM

Let  $\Gamma$  be a closed (compact and without a boundary) infinitely smooth manifold of dimension  $n \geq 1$ . We suppose that a certain  $C^\infty$ -density  $dx$  is defined on  $\Gamma$ . By  $\mathcal{D}'(\Gamma)$  we denote the linear topological space of all distributions on  $\Gamma$ , that is  $\mathcal{D}'(\Gamma)$  is a space antidual to the space  $C^\infty(\Gamma)$  with respect to the extension of the scalar product in  $L_2(\Gamma, dx)$  by continuity. This extension is denoted by  $(f, w)_\Gamma$  for  $f \in \mathcal{D}'(\Gamma)$ ,  $w \in C^\infty(\Gamma)$ .

We consider a system of linear equations

$$\sum_{k=1}^p A_{j,k} u_k = f_j \quad \text{on } \Gamma, \quad j = 1, \dots, p. \quad (2)$$

Here,  $p \in \mathbb{N}$  and  $A_{j,k}$ ,  $j, k = 1, \dots, p$ , are scalar classical (polyhomogeneous) pseudodifferential operators of arbitrary real orders defined on the manifold  $\Gamma$  (see e.g. [2, Sec. 2.1]). A complete symbol of the pseudodifferential operator  $A_{j,k}$  is an infinitely smooth complex-valued function on the cotangent bundle  $T^*\Gamma$ . A principal symbol of  $A_{j,k}$  which is positively homogeneous of order  $\text{ord } A_{j,k}$  in every section  $T_x^*\Gamma \setminus \{0\}$ ,  $x \in \Gamma$  and, moreover, is not identically equal to zero, is also defined. We consider equations (2) in the sense of the distribution theory, so  $u_k, f_j \in \mathcal{D}'(\Gamma)$ . We put for every index  $k = 1, \dots, p$

$$m_k := \max \{ \text{ord } A_{1,k}, \dots, \text{ord } A_{p,k} \}.$$

Let us assume system (2) to be *elliptic in Petrovskii's sense*, that is

$$\det \left( a_{j,k}^{(0)}(x, \xi) \right)_{j,k=1}^p \neq 0 \quad \text{for each } x \in \Gamma, \xi \in T_x^*\Gamma \setminus \{0\}.$$

Here  $a_{j,k}^{(0)}(x, \xi)$  is the principal symbol of the pseudodifferential operator  $A_{j,k}$  in the case  $\text{ord } A_{j,k} = m_k$ , or  $a_{j,k}^{(0)}(x, \xi) \equiv 0$  in the case  $\text{ord } A_{j,k} < m_k$ .

Let us rewrite system (2) in the matrix form:  $Au = f$  on  $\Gamma$ , where  $A := (A_{j,k})$  is a square matrix of order  $p$ , and  $u = \text{col}(u_1, \dots, u_p)$ ,  $f = \text{col}(f_1, \dots, f_p)$  are functional columns. The mapping  $u \mapsto Au$  is a linear continuous operator in the space  $(\mathcal{D}'(\Gamma))^p$ .

## 3. A REFINED SCALE OF SPACES

We denote by  $\mathcal{M}$  the set of all Borel measurable functions  $\varphi : [1, +\infty) \rightarrow (0, +\infty)$  such that the functions  $\varphi$  and  $1/\varphi$  are bounded on every closed interval  $[1, b]$ , where  $1 < b < +\infty$ , and the function  $\varphi$  is slowly varying at  $+\infty$  in Karamata's sense, that is

$$\lim_{t \rightarrow +\infty} \varphi(\lambda t) / \varphi(t) = 1 \quad \text{for each } \lambda > 0.$$

Let  $s \in \mathbb{R}$ ,  $\varphi \in \mathcal{M}$ . We denote by  $H^{s,\varphi}(\mathbb{R}^n)$  the set of all tempered distributions  $u$  such that the Fourier transform  $\widehat{u}$  of the distribution  $u$  is a function locally Lebesgue integrable in  $\mathbb{R}^n$  which satisfies the condition

$$\int \langle \xi \rangle^{2s} \varphi^2(\langle \xi \rangle) |\widehat{u}(\xi)|^2 d\xi < \infty.$$

Here the integral is evaluated over  $\mathbb{R}^n$ , and  $\langle \xi \rangle := (1 + \xi_1^2 + \dots + \xi_n^2)^{1/2}$ . In the space  $H^{s,\varphi}(\mathbb{R}^n)$  we define the inner product

$$(u, v)_{H^{s,\varphi}(\mathbb{R}^n)} := \int \langle \xi \rangle^{2s} \varphi^2(\langle \xi \rangle) \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi.$$

The space  $H^{s,\varphi}(\mathbb{R}^n)$  is a special isotropic Hilbert case of the spaces introduced by L. Hörmander [3, Sec. 2.2], [4, Sec. 10.1] and L. R. Volevich, B. P. Paneah [5, Sec. 2], [6, Sec. 1.4.2]. In the simplest case where  $\varphi(\cdot) \equiv 1$  the space  $H^{s,\varphi}(\mathbb{R}^n)$  coincides with the Sobolev space. It follows from the inclusions

$$\bigcup_{\varepsilon > 0} H^{s+\varepsilon}(\mathbb{R}^n) =: H^{s+}(\mathbb{R}^n) \subset H^{s,\varphi}(\mathbb{R}^n) \subset H^{s-}(\mathbb{R}^n) := \bigcap_{\varepsilon > 0} H^{s-\varepsilon}(\mathbb{R}^n)$$

that in the collection of spaces

$$\{H^{s,\varphi}(\mathbb{R}^n) : s \in \mathbb{R}, \varphi \in \mathcal{M}\} \quad (3)$$

the functional parameter  $\varphi$  defines an additional (subpower) smoothness with respect to the basic (power)  $s$ -smoothness. Otherwise speaking,  $\varphi$  *refines* the power smoothness.

The *refined scale* over the manifold  $\Gamma$  is constructed from scale (3) in the usual way. Let us take a finite atlas from the  $C^\infty$ -structure on  $\Gamma$  consisting of the local charts  $\alpha_j : \mathbb{R}^n \leftrightarrow U_j$ ,  $j = 1, \dots, r$ . Here the open sets  $U_j$  form the finite covering of the manifold  $\Gamma$ . Let functions  $\chi_j \in C^\infty(\Gamma)$ ,  $j = 1, \dots, r$ , form a partition of unity on  $\Gamma$  satisfying the condition  $\text{supp } \chi_j \subset U_j$ .

Let us set

$$H^{s,\varphi}(\Gamma) := \{h \in \mathcal{D}'(\Gamma) : (\chi_j h) \circ \alpha_j \in H^{s,\varphi}(\mathbb{R}^n) \text{ for every } j = 1, \dots, r\}.$$

Here  $(\chi_j h) \circ \alpha_j$  is the representation of the distribution  $\chi_j h$  in the local chart  $\alpha_j$ . The inner product in the space  $H^{s,\varphi}(\Gamma)$  is defined by the formula

$$(f, g)_{H^{s,\varphi}(\Gamma)} := \sum_{j=1}^r ((\chi_j f) \circ \alpha_j, (\chi_j g) \circ \alpha_j)_{H^{s,\varphi}(\mathbb{R}^n)}$$

and induces the norm  $\|h\|_{s,\varphi} := (h, h)_{s,\varphi}^{1/2}$ .

The Hilbert space  $H^{s,\varphi}(\Gamma)$  is separable, continuously imbedded into the space  $\mathcal{D}'(\Gamma)$ , and independent (up to equivalent norms) of the choice of the atlas and the partition of unity. The collection of function spaces

$$\{H^{s,\varphi}(\Gamma) : s \in \mathbb{R}, \varphi \in \mathcal{M}\} \quad (4)$$

is naturally called the refined scale over the manifold  $\Gamma$ .

This scale admits an alternative intrinsic description. Let the Riemannian structure on the manifold  $\Gamma$  which defines the density  $dx$  be given (it is always possible), and let  $\Delta_\Gamma$  be the Beltrami-Laplace operator on  $\Gamma$ . For  $s \in \mathbb{R}$  and  $\varphi \in \mathcal{M}$ , we define the function

$$\varphi_s(t) := t^{s/2} \varphi(t^{1/2}) \text{ for } t \geq 1 \quad \text{and} \quad \varphi_s(t) := \varphi(1) \text{ for } 0 < t < 1.$$

We consider the operator  $\varphi_s(1 - \Delta_\Gamma)$  in the space  $L_2(\Gamma, dx)$  as a Borel function of the self-adjoint operator  $1 - \Delta_\Gamma$ .

**Proposition 1.** *For arbitrary  $s \in \mathbb{R}$ ,  $\varphi \in \mathcal{M}$ , the space  $H^{s,\varphi}(\Gamma)$  coincides with the completion of the set of functions  $u \in C^\infty(\Gamma)$  with respect to the norm  $\|\varphi_s(1 - \Delta_\Gamma)u\|_{L_2(\Gamma, dx)}$  which is equivalent to the norm  $\|u\|_{s,\varphi}$ .*

The following refinement of the classical Sobolev theorem characterizes separating possibilities of scale (4).

**Proposition 2.** *Let a function  $\varphi \in \mathcal{M}$  and an integer  $\rho \geq 0$  be given. The inequality*

$$\int_1^{+\infty} \frac{dt}{t \varphi^2(t)} < \infty \quad (5)$$

*is equivalent to the continuous imbedding  $H^{\rho+n/2,\varphi}(\Gamma) \hookrightarrow C^\rho(\Gamma)$ . The continuity of this imbedding implies its compactness.*

#### 4. THE BASIC RESULTS

We denote by  $A^+$  a matrix pseudodifferential operator formally adjoint to  $A$  with respect to the form  $(\cdot, \cdot)_\Gamma$ . We set

$$N := \{u \in (C^\infty(\Gamma))^p : Au = 0 \text{ on } \Gamma\} \quad \text{and} \quad N^+ := \{v \in (C^\infty(\Gamma))^p : A^+v = 0 \text{ on } \Gamma\}.$$

The ellipticity of system (2) implies that the spaces  $N$  and  $N^+$  are finite-dimensional [2, c. 52].

**Theorem 1.** *For each  $s \in \mathbb{R}$ ,  $\varphi \in \mathcal{M}$  the linear bounded operator*

$$A : \prod_{k=1}^p H^{s+m_k,\varphi}(\Gamma) \rightarrow (H^{s,\varphi}(\Gamma))^p \quad (6)$$

*is defined. It is a Fredholm one, has the kernel  $N$  and the range*

$$\left\{ f \in (H^{s,\varphi}(\Gamma))^p : \sum_{j=1}^p (f_j, w_j)_\Gamma = 0 \text{ for each } (w_1, \dots, w_p) \in N^+ \right\}.$$

*The index of the operator (6) is equal to  $\dim N - \dim N^+$  and is independent of  $s, \varphi$ .*

According to this theorem,  $N^+$  is the defect subspace of operator (6). Let's note [19], [2, p. 32] that in the scalar case ( $p = 1$ ) the index of operator (6) is equal to 0 if  $\dim \Gamma \geq 2$ . Another sufficient condition for this property is the ellipticity of system with a parameter on a certain ray  $K := \{\lambda \in \mathbb{C} : \arg \lambda = \text{const}\}$  [2].

**Theorem 2.** *For arbitrarily chosen parameters  $s \in \mathbb{R}$ ,  $\varphi \in \mathcal{M}$  and  $\sigma > 0$ , the following a priori estimate holds:*

$$\sum_{k=1}^p \|u_k\|_{s+m_k,\varphi} \leq c \left( \sum_{j=1}^p \|f_j\|_{s,\varphi} + \sum_{k=1}^p \|u_k\|_{s-\sigma} \right).$$

*Here the number  $c > 0$  is independent of vector-functions  $u$ ,  $f = Au$ .*

If the spaces  $N$  and  $N^+$  are trivial, then operator (6) is a topological isomorphism. Generally, it is convenient to define the isomorphism with the help of two projectors. Let's consider the spaces in which operator (6) acts. Let us decompose them in the following direct sums of the closed subspaces:

$$\prod_{k=1}^p H^{s+m_k, \varphi}(\Gamma) = N \dot{+} \left\{ u : \sum_{k=1}^p (u_k, v_k)_\Gamma = 0 \text{ for each } (v_1, \dots, v_p) \in N \right\},$$

$$(H^{s, \varphi}(\Gamma))^p = N^+ \dot{+} A((H^{s, \varphi}(\Gamma))^p).$$

We denote by  $P$  and  $P^+$  respectively the oblique projectors of these spaces onto the second terms in the sums. The projectors are independent of  $s, \varphi$ .

**Theorem 3.** *For arbitrary  $s \in \mathbb{R}$ ,  $\varphi \in \mathcal{M}$ , the restriction of operator (6) onto the subspace  $P(\prod_{k=1}^p H^{s+m_k, \varphi}(\Gamma))$  establishes the topological isomorphism*

$$A : P\left(\prod_{k=1}^p H^{s+m_k, \varphi}(\Gamma)\right) \leftrightarrow P^+((H^{s, \varphi}(\Gamma))^p).$$

## 5. AN APPLICATION

Let  $\Gamma_0$  be an open nonempty subset of the manifold  $\Gamma$ . Denote

$$H_{\text{loc}}^{s, \varphi}(\Gamma_0) := \{f \in \mathcal{D}'(\Gamma) : \chi f \in H^{s, \varphi}(\Gamma) \text{ for each } \chi \in C^\infty(\Gamma), \text{ supp } \chi \subset \Gamma_0\}.$$

**Theorem 4.** *Suppose that a vector-function  $u \in (\mathcal{D}'(\Gamma))^p$  is a solution of the equation  $Au = f$  on the set  $\Gamma_0$ , where  $f \in (H_{\text{loc}}^{s, \varphi}(\Gamma_0))^p$  for some parameters  $s \in \mathbb{R}$  and  $\varphi \in \mathcal{M}$ . Then  $u \in \prod_{k=1}^p H_{\text{loc}}^{s+m_k, \varphi}(\Gamma_0)$ .*

This theorem specifies, with regard to refined scale (4), known propositions on local lifting of interior smoothness of an elliptic system solution in the Sobolev scale (see e.g. [20, 3, 21]). Note that the refined local smoothness  $\varphi$  of the right-hand side of the elliptic system is inherited by its solution. Theorem 4 and Proposition 1 imply the following sufficient condition for a chosen component  $u_k$  of the solution of system (2) to have continuous derivatives of a prescribed order.

**Corollary 1.** *Suppose that vector-functions  $u, f \in (\mathcal{D}'(\Gamma))^p$  satisfy the equation  $Au = f$  on  $\Gamma_0$ . Let integers  $\rho \geq 0$ ,  $k = 1, \dots, p$ , and a function  $\varphi \in \mathcal{M}$  be such that inequality (5) is true. Then*

$$\left( f_j \in H_{\text{loc}}^{\rho-m_k+n/2, \varphi}(\Gamma_0) \text{ for every } j = 1, \dots, p \right) \Rightarrow u_k \in C^\rho(\Gamma_0).$$

## REFERENCES

- [1] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Vol. 3*, Springer-Verlag, Berlin etc, 1985.
- [2] M. S. Agranovich, *Partial differential equations. VI. Elliptic operators on closed manifolds*, Encycl. Math. Sci. **63**, Springer-Verlag, Berlin etc, 1994, 1–130.
- [3] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin etc, 1963.
- [4] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Vol. 2*, Springer-Verlag, Berlin etc, 1983.

- [5] L. R. Volevich, B. P. Paneah, *Certain spaces of generalized functions and imbedding theorems*, Usp. Mat. Nauk. **20** (1965), no. 1, 3–74. (Russian)
- [6] B. Paneah, *The Oblique Derivative Problem. The Poincaré Problem*, Wiley – VCH, Berlin etc, 2000.
- [7] V. A. Mikhailets, A. A. Murach, *Improved scale of spaces and elliptic boundary-value problems. I*, Ukr. math. J. **58** (2006), no. 2, 244–262.
- [8] V. A. Mikhailets, A. A. Murach, *Improved scale of spaces and elliptic boundary-value problems. II*, Ukr. math. J. **58** (2006), no. 3, 398–417.
- [9] Sh. A. Alimov, V. A. Il'in, E. M. Nikishin, *Convergence problems of multiple trigonometric series and spectral decompositions. I*, Russ. Math. Surv. **31** (1976), no. 6, 29–86.
- [10] V. A. Mikhailets, *Asymptotics of the spectrum of elliptic operators and boundary conditions*, Sov. Math., Dokl. **26** (1982), no. 5, 464–468.
- [11] V. A. Mikhailets, *A precise estimate of the remainder in the spectral asymptotics of general elliptic boundary problems*, Funct. Anal. Appl. **23** (1989), no. 2, 137–139.
- [12] G. A. Kalyabin, P. I. Lizorkin, *Spaces of functions of generalized smoothness*, Math. Nachr., **133** (1987), 7–32.
- [13] D. D. Haroske, S. D. Moura, *Continuity envelopes of spaces of generalised smoothness, entropy and approximation numbers*, J. Approximation Theory, **128** (2004), 151–174.
- [14] W. Farkas, H.-G. Leopold, *Characterisations of function of generalized smoothness*, Ann. Mat. Pura Appl. **185** (2006), no. 1, 1–62.
- [15] V. A. Mikhailets, A. A. Murach, *Refined scale of spaces and elliptic boundary-value problems. III*, Ukr. math. J. **59** (2007), no. 5, 679–701. (Russian)
- [16] V. A. Mikhailets, A. A. Murach, *An elliptic operator in the refined scale of spaces on a closed manifold*, Dopov. Nats. Acad. Nauk. Ukr., Mat. Pryr. Tehn. Nauky (2006), no. 10, 27–33. (Russian)
- [17] V. A. Mikhailets, A. A. Murach, *Regular elliptic boundary-value problem for homogeneous equation in two-sided refined scale of spaces*, Ukr. Math. J. **58** (2006), no. 11, 1536–1555. (Russian)
- [18] V. A. Mikhailets, A. A. Murach, *Elliptic operator with homogeneous regular boundary conditions in two-sided refined scale of spaces*, Ukr. Math. Bull **3** (2006), no. 4, 529–560.
- [19] M. F. Atiyah, I. M. Singer, *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc., **69** (1963), no. 3, 422–433.
- [20] A. Douglis, L. Nirenberg, *Interior estimates for elliptic systems of partial differential equations*, Commun. Pure Appl. Math., **8** (1955), no. 4, 503–538.
- [21] Yu. M. Berezanskij, *Expansions in Eigenfunctions of Selfadjoint Operators*, Transl. Math. Monographs **17**, Am. Math. Soc., Providence, 1968.

INSTITUTE OF MATHEMATICS NAS OF UKRAINE, TERESHCHENKIVSKA STR., 3, KYIV, UKRAINE, 01601  
*E-mail address*: mikhailets@imath.kiev.ua, murach@imath.kiev.ua